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A note on the singularity of an inner problem for head-sea diffraction by a slender body

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In studying head-sea diffraction by a slender body via matched asymptotics, Haren & Mei (1981) found numerically that the normalized inner problem is singular at a special frquency. In the present note, this singularity is confirmed by a different numerical method and by an analytical demonstration in the shallow-water limit.

1. Introduction

In a recent paper by Haren & Mei (1980, hereafter referred to as I), a singularity is encountered in the inner problem of the matched asymptotic method for a slender zero-draft body in head seas in deep water. For arbitrary depth h, the decomposition for the inner problem is

$$\phi = e^{ikx}(\cosh k(z+h)+\psi), \tag{1}$$

where ϕ is the total potential and the diffraction potential ψ satisfies

$$\psi_{yy}+\psi_{zz}-k^2\psi = 0, \quad -h < z < 0; \tag{2a}$$

$$\psi_z-\sigma\psi = 0 \quad \text{on} \quad z = 0 \quad (\sigma = k\tanh kh); \tag{2b}$$

$$\psi_z = 0 \quad \text{on} \quad z = -h; \tag{2c}$$

$$\frac{\partial\psi}{\partial n} = -\frac{\partial}{\partial n}(\cosh k(z+h)) \quad \text{on body } B; \tag{2d}$$

$$\psi \to A_0^{\pm} y\cosh k(z+h), \quad ky \to \pm\infty. \tag{2e}$$

The purpose of this note is to provide further evidence that the singularity is inherent in the inner mathematical problem as posed, and is independent of the technique of solution, unlike the well-known *irregular frequencies* in diffraction theory. Indeed, for the geometry of I of a zero-draft body of width b, a unique frequency $(kb)^*$ ($(kb)^* = O(1)$) can always be found for *any water depth* near which the solution behaves like $((kb)^*-(kb))^{-1}$. For long waves in shallow water ($kh \ll 1$) such a singularity is first shown analytically, while, in the case of arbitrary depth, numerical evidence is provided using a completely different method to that of I. A similar inner problem was addressed by Ursell (1968), but he proved uniqueness only in the limit $k \to 0$.

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2. Shallow-water limit

We invoke a shallow-water approximation in (2) for a flat-bottomed body with any clearance H $(0 < H \leqslant h)$. Expanding ψ in terms of Taylor series in $z+h$, it is easily found that

$$\psi(y,z) = \psi_0(y) + \tfrac{1}{2}(k(z+h))^2\psi_2(y) + O[(kh)^4] \tag{3}$$

and ψ_0 satisfies, to leading order,

$$\psi_0'' - k^2\psi_0 = k^2, \quad |y| < \tfrac{1}{2}b, \tag{4}$$

and

$$\bar{\psi}_0'' = 0, \quad |y| > \tfrac{1}{2}b. \tag{5}$$

where the overbar denotes quantities for $|y| > \frac{1}{2}b$. Under the raft, (4) has the symmetric solution

$$\psi_0 = C\cosh ky - 1, \quad |y| < \tfrac{1}{2}b, \tag{6}$$

while, outside, we have from (5),

$$\bar{\psi}_0 = A_0|y|, \quad |y| > \tfrac{1}{2}b, \tag{7}$$

and no constant term is admitted on account of (2e). Invoking the continuity of pressure ($\bar{\psi}_0 = \psi_0$) and flux ($h\bar{\psi}_0' = H\psi_0'$) at $|y| = \frac{1}{2}b$, we obtain a pair of inhomogeneous algebraic equations for A_0 and C. Non-trivial homogeneous solutions exist when the coefficient determinant vanishes

$$\tfrac{1}{2}kb\tanh(\tfrac{1}{2}kb) - h/H = 0. \tag{8}$$

Now this equation possesses a single positive root $(kb)^*$ so that the solution to the inner problem has a simple pole there. Away from that frequency, the solution is easily found.

3. Arbitrary finite depth

In this case, a simple explicit solution is not available even for a body with zero draft, and we solve the inner problem numerically using a variational principle, which is rather different from the integral equation method of I. Because of the simple geometry, we represent both the solution ψ underneath the raft ($|y| < \frac{1}{2}b$), and $\bar{\psi}$ outside ($|y| > \frac{1}{2}b$) by separate analytic series. Let

$$\psi(y,z) = \psi_p(z) + \psi_1(y,z), \quad |y| < \tfrac{1}{2}b, \tag{9}$$

where

$$\psi_p = -\cosh k(z+h) \tag{10}$$

is the particular solution satisfying the forcing on $z = 0$ (equation (2d)) and ψ_1 is the eigenfunction series with zero surface flux

$$\psi_1 = \sum_{m=0} C_m \cos\frac{m\pi(z+h)}{h}\cosh[k^2 + (m\pi/h)^2]^{\frac{1}{2}}y. \tag{11}$$

For $|y| > \frac{1}{2}b$, we use the representation

$$\bar{\psi} = A_0|y|\cosh k(z+h) + \sum_{n=1} A_n \exp[-(k^2-k_n^2)^{\frac{1}{2}}|y|]\cos k_n(z+h), \quad |y| > \tfrac{1}{2}b, \tag{12}$$

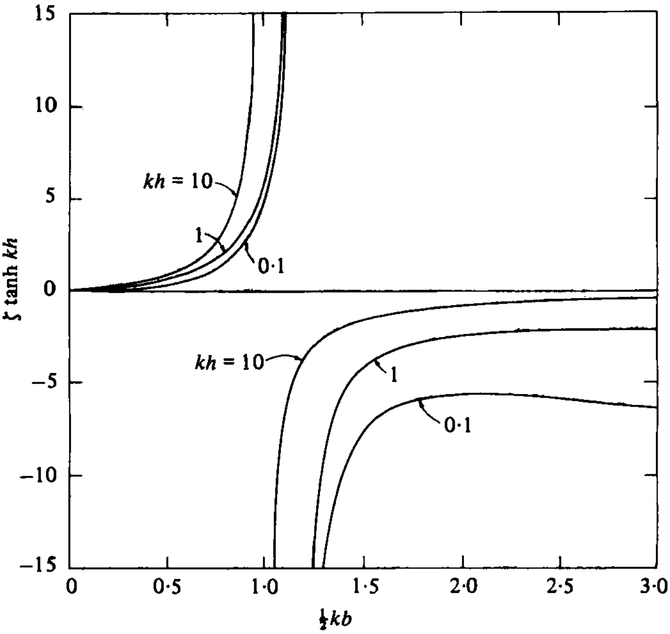

FIGURE 1

where k_n are the positive real solutions of $k \tan kh = -\sigma$. The boundary-value problem for $(\psi, \overline{\psi})$ would be completely satisfied if we further require the matching conditions

$$\left.\begin{aligned} \psi - \overline{\psi} &= 0 \\ \text{and} \qquad (\psi - \overline{\psi})_y &= (\psi_1 - \overline{\psi})_y = 0 \quad \text{on} \quad |y| = \tfrac{1}{2}b. \end{aligned}\right\} \tag{13}$$

It can be shown that equations (13) are equivalent to the stationarity of the following functional:

$$J(\psi, \overline{\psi}) = \int_{-h}^{0} dz\,(\overline{\psi}\,\overline{\psi}_y + \psi_1 \psi_{1y} - 2\psi \overline{\psi}_y)|_{y=\frac{1}{2}b}. \tag{14}$$

To see this, we take the first variation of (14) and obtain after collecting terms

$$\begin{aligned} \delta J = 2\int_{-h}^{0} & dz[\delta\overline{\psi}_y(\overline{\psi} - \psi) + \delta\psi_1(\psi_{1y} - \overline{\psi}_y)] \\ & + \int_{-h}^{0} dz[(\delta\overline{\psi}\,\overline{\psi}_y - \overline{\psi}\delta\overline{\psi}_y) + (\psi_1 \delta\psi_{1y} - \delta\psi_1 \psi_{1y})], \end{aligned} \tag{15}$$

where (9) is used and $\delta\psi_p \equiv 0$ since ψ_p is known. The second integral in (15) can be shown to vanish by applying Green's identity separately to $\overline{\psi}$, $\delta\overline{\psi}$ and ψ_1, $\delta\psi_1$ respectively in the domains $|y| >, < \frac{1}{2}b$. Hence $\delta J = 0$ if and only if $(\psi, \overline{\psi})$ satisfy (13). Substituting the truncated series for ψ_1, $\overline{\psi}$ into (14), the vanishing of the first variation of J reduces to a system of symmetric algebraic equations for A_n and C_n, which can be readily inverted.

The results are shown in figure 1 for $kh = 0{\cdot}1$, 1 and 10, where ζ is the normalized force acting on a cross-section as defined in equation (4.1) of I. The curves for $kh = 0{\cdot}1$ and 10 are respectively indistinguishable from the shallow-water results and that of figure 7 of I, confirming the accuracy of the present numerical method. By extrapolation, the singular frequencies for $kh = 0{\cdot}1$, 1, 10 are determined to be respectively at $\frac{1}{2}kb = 1{\cdot}200$, $1{\cdot}170$ and $0{\cdot}993$, in the neighbourhood of which the matrix equation resulting from (14) becomes numerically singular. By examining the immediate neighbourhood numerically, it is also found that the singularity is a simple pole. We conclude that the normalized inner problem has unbounded solution at a single frequency for any arbitrary depth.

4. Concluding remarks

The occurrence of a singular solution to the boundary-value problem (2) at a specific frequency reminds one of the phenomenon of *irregular frequencies* encountered in diffraction theory when the method of integral equation is used. The solutions of the Fredholm integral equation obtained by distributing sources over the body breaks down at discrete frequencies which correspond to the eigenfrequencies of a fictitious interior problem. This failure, however, is a property of the integral equation method and not the original boundary-value problem which is unique. Such singularities, for example, are shown to be absent when an alternative method based on a variational formulation is used (Aranha, Mei & Yue 1979). In the present case, an associated interior problem does not exist for the integral equation method. Our evidence shows that the singularity is present in the original boundary-value problem and is not the peculiarity of a solution method.

The present difficulty is a result of the decomposition (1) which leads into the inner problem for ψ (cf. equation (2.7) of I). In particular, the asymptotic value of ψ for $|y| \gg b$ no longer contains the homogenous solution $\cosh k(z+h)$ or a constant term in (7) for shallow water. As was pointed out in I, the solution to the integral equation resulting from matching with the outer solution (equation (2.34) of I) produces no difficulty in principle across the singularity. The whole problem of an unbounded inner solution can in fact be avoided completely if the inner problem can be posed in terms of the total potential satisfying zero normal velocity on the body. This is already known for the special case of shallow water (Mei & Tuck 1980). Such a decomposition, however, cannot be avoided in a general formulation of head-sea diffraction using matched asymptotic method where the inner problem must in principle be solved numerically before matching.

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